

# Cliques and the Spectral Radius

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## Abstract

We prove a number of relations between the number of cliques of a graph  $G$  and the largest eigenvalue  $\mu(G)$  of its adjacency matrix. In particular, writing  $k_s(G)$  for the number of  $s$ -cliques of  $G$ , we show that, for all  $r \geq 2$ ,

$$\mu^{r+1}(G) \leq (r+1) k_{r+1}(G) + \sum_{s=2}^r (s-1) k_s(G) \mu^{r+1-s}(G),$$

and, if  $G$  is of order  $n$ , then

$$k_{r+1}(G) \geq \left( \frac{\mu(G)}{n} - 1 + \frac{1}{r} \right) \frac{r(r-1)}{r+1} \left( \frac{n}{r} \right)^{r+1}.$$

**Keywords:** *number of cliques, clique number, spectral radius, stability*

## 1 Introduction

Our graph-theoretic notation is standard (e.g., see [1]); in particular, we write  $G(n)$  for a graph of order  $n$ . Given a graph  $G$ , a  $k$ -walk is a sequence of vertices  $v_1, \dots, v_k$  of  $G$  such that  $v_{i-1}$  is adjacent to  $v_i$  for all  $i = 2, \dots, k$ . We write  $w_k(G)$  for the number of  $k$ -walks in  $G$  and  $k_r(G)$  for the number of its  $r$ -cliques. We order the eigenvalues of the adjacency matrix of a graph  $G = G(n)$  as  $\mu(G) = \mu_1(G) \geq \dots \geq \mu_n(G)$ .

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Let  $\omega = \omega(G)$  be the clique number of  $G$ . Wilf [12] proved that

$$\mu(G) \leq \frac{\omega - 1}{\omega} v(G) = \frac{\omega - 1}{\omega} w_1(G),$$

and Nikiforov [9] extended this, showing that the inequality

$$\mu^s(G) \leq \frac{\omega - 1}{\omega} w_s(G) \tag{1}$$

holds for every  $s \geq 1$ . Note that for  $s = 2$  inequality (1) implies a concise form of Turán's theorem. Indeed, if  $G$  has  $n$  vertices and  $m$  edges, then  $\mu(G) \geq 2m/n$ , and so,

$$\left(\frac{2m}{n}\right)^2 \leq \mu^2(G) \leq \frac{\omega - 1}{\omega} w_2(G) = \frac{\omega - 1}{\omega} 2m.$$

This shows that

$$m \leq \frac{\omega - 1}{2\omega} n^2, \tag{2}$$

which is best possible whenever  $\omega$  divides  $n$ . If we combine (1) with other lower bounds on  $\mu(G)$ , e.g., with

$$\mu^2(G) \geq \frac{1}{n} \sum_{u \in V(G)} d^2(u),$$

we obtain generalizations of (2).

Moreover, inequality (1) follows from a result of Motzkin and Straus [7] following in turn from (2) (see [10]). The implications

$$(1) \implies (2) \implies \text{MS} \implies (1)$$

justify regarding inequality (1) as a spectral form of Turán's theorem, well suited for nontrivial generalizations. For example, the following conjecture seems to be quite subtle.

**Conjecture 1** *Let  $G$  be a  $K_{r+1}$ -free graph with  $m$  edges. Then*

$$\mu_1^2(G) + \mu_2^2(G) \leq \frac{r - 1}{r} 2m.$$

If true, this conjecture is best possible whenever  $r$  divides  $n$ . Indeed, for  $r|n$ ,  $n = qr$ , the Turán graph  $T_r(n)$  (i.e., the complete  $r$ -partite graph  $K_r(q)$  with  $q$  vertices in each class) has  $r(r - 1)q^2/2$  edges, and there are three eigenvalues:  $(r - 1)q$ , with multiplicity 1,  $-q$ , with multiplicity  $r - 1$ , and 0, with multiplicity  $r(q - 1)$ , so that  $\mu_1(G) = (r - 1)q$  and  $\mu_2(G) = 0$ .

The aim of this note is to prove further relations between  $\mu(G)$  and the number of cliques in  $G$ . In [8] it is proved that

$$\mu^\omega(G) \leq \sum_{s=2}^{\omega} (s-1) k_s(G) \mu^{\omega-s}(G) \quad (3)$$

with equality holding if and only if  $G$  is a complete  $\omega$ -partite graph with possibly some isolated vertices. It turns out that this inequality is one of a whole sequence of similar inequalities.

**Theorem 1** *For every graph  $G$  and  $r \geq 2$ ,*

$$\mu^{r+1}(G) \leq (r+1) k_{r+1}(G) + \sum_{s=2}^r (s-1) k_s(G) \mu^{r+1-s}(G).$$

Observe that, with  $r = \omega + 1$ , Theorem 1 implies (3). Theorem 1 also implies a lower bound on the number of cliques of any given order, as stated below.

**Theorem 2** *For every graph  $G = G(n)$  and  $r \geq 2$ ,*

$$k_{r+1}(G) \geq \left( \frac{\mu(G)}{n} - 1 + \frac{1}{r} \right) \frac{r(r-1)}{r+1} \left( \frac{n}{r} \right)^{r+1}.$$

We also prove the following extension of an earlier result of ours [2].

**Theorem 3** *Let  $1 \leq s \leq r < \omega(G)$  and  $\alpha \geq 0$ . If  $G = G(n)$  and*

$$(s+1) k_{s+1}(G) \geq n^{s+1} \prod_{t=1}^s \left( \frac{r-t}{rt} + \alpha \right), \quad (4)$$

*then*

$$k_{r+1}(G) \geq \alpha \frac{r^2}{r+1} \left( \frac{n}{r} \right)^{r+1}. \quad (5)$$

Note that Theorems 3 and 2 hold for all values of the parameters satisfying the conditions there; in particular,  $\alpha$  may depend on  $n$ .

Our final theorem is the following stability result.

**Theorem 4** For all  $r \geq 2$  and  $0 \leq \alpha \leq 2^{-10}r^{-6}$ , if  $G = G(n)$  is a  $K_{r+1}$ -free graph with

$$\mu(G) \geq \left(1 - \frac{1}{r} - \alpha\right)n, \quad (6)$$

then  $G$  contains an induced  $r$ -partite graph  $G_0$  of order  $v(G_0) > (1 - 3\alpha^{1/3})n$  and minimum degree

$$\delta(G_0) > \left(1 - \frac{1}{r} - 6\alpha^{1/3}\right)n.$$

## 2 Proofs

### 2.1 Proof of Theorem 1

For a vertex  $u \in V(G)$ , write  $w_l(u)$  for the number of  $l$ -walks starting with  $u$  and  $k_r(u)$  for the number of  $r$ -cliques containing  $u$ . Clearly, it is enough to prove the assertion for  $2 \leq r < \omega(G)$ , since the case  $r \geq \omega(G)$  follows easily from (3).

It is shown in [8] that for all  $2 \leq s \leq \omega(G)$  and  $l \geq 2$ ,

$$\sum_{u \in V(G)} (k_s(u) w_{l+1}(u) - k_{s+1}(u) w_l(u)) \leq (s-1) k_s(G) w_l(G). \quad (7)$$

Summing these inequalities for  $s = 2, \dots, r$ , we obtain

$$\sum_{u \in V(G)} (k_2(u) w_{l+r-1}(u) - k_{r+1}(u) w_l(u)) \leq \sum_{s=2}^r (s-1) k_s(G) w_{l+r-s}(G),$$

and so, after rearranging,

$$w_{l+r}(G) - \sum_{s=2}^r (s-1) k_s(G) w_{l+r-s}(G) \leq \sum_{u \in V(G)} k_{r+1}(u) w_l(u).$$

Noting that  $w_l(u) \leq w_{l-1}(G)$ , this implies that

$$\sum_{u \in V(G)} k_{r+1}(u) w_l(u) \leq w_{l-1}(G) \sum_{u \in V(G)} k_{r+1}(u) = (r+1) k_{r+1}(G) w_{l-1}(G),$$

and so,

$$\frac{w_{l+r}(G)}{w_{l-1}(G)} - \sum_{s=2}^r (s-1) k_s(G) \frac{w_{l+r-s}(G)}{w_{l-1}(G)} \leq (r+1) k_{r+1}(G).$$

Given  $n$ , there are non-negative constants  $c_1, \dots, c_n$  such that for  $G = G(n)$  we have

$$w_l(G) = c_1 \mu_1^{l-1}(G) + \dots + c_n \mu_n^{l-1}(G),$$

(See, e.g., [3], p. 44.) Since  $\omega > 2$ , our graph  $G$  is not bipartite and so  $|\mu_n(G)| < \mu_1(G)$ . Therefore, for every fixed  $q$ , we have

$$\lim_{l \rightarrow \infty} \frac{w_{l+q}(G)}{w_{l-1}(G)} = \mu^{q+1}(G),$$

and the assertion follows.  $\square$

## 2.2 Proof of Theorem 3

Moon and Moser [6] stated the following result (for a proof see [4] or [5], Problem 11.8): if  $G = G(n)$  and  $k_s(G) > 0$ , then

$$\frac{(s+1)k_{s+1}(G)}{sk_s(G)} - \frac{n}{s} \geq \frac{sk_s(G)}{(s-1)k_{s-1}(G)} - \frac{n}{s-1}.$$

Equivalently, for  $1 \leq s < t < \omega(G)$ , we have

$$\frac{(t+1)k_{t+1}(G)}{tk_t(G)} - \frac{n}{t} \geq \frac{(s+1)k_{s+1}(G)}{sk_s(G)} - \frac{n}{s}. \quad (8)$$

Let  $s \in [1, r]$  be the smallest integer for which (4) holds. This implies either  $s = 1$  or

$$sk_s(G) < n^s \prod_{t=1}^{s-1} \left( \frac{r-t}{rt} + \alpha \right) \quad (9)$$

for some  $s \in [2, r]$ . Suppose first that  $s = 1$ . (This case is considered in [2], but for the sake of completeness we present it here.) We have

$$\frac{2k_2(G)}{k_1(G)} - n \geq \left( \frac{r-1}{r} + \alpha \right) n - n = \alpha n - \frac{n}{r},$$

and so, for all  $t = 1, \dots, r$ , inequality (8) implies that

$$\frac{(t+1)k_{t+1}(G)}{tk_t(G)} \geq \alpha n + \frac{n}{t} - \frac{n}{r}.$$

Multiplying these inequalities for  $t = 1, \dots, r$ , we obtain that

$$(r+1)k_{r+1}(G) \geq n^{r+1} \prod_{t=1}^r \left( \frac{r-t}{rt} + \alpha \right) \geq \alpha r^2 \left( \frac{n}{r} \right)^{r+1} \prod_{t=1}^{r-1} \frac{r-t}{t} = \alpha r^2 \left( \frac{n}{r} \right)^{r+1},$$

proving the result in this case.

Assume now that (9) holds for some  $s \in [2, r]$ . Then we have

$$\frac{(s+1)k_{s+1}(G)}{sk_s(G)} > \left( \frac{r-s}{rs} + \alpha \right) n.$$

and so, for every  $t = s, \dots, r$ ,

$$\frac{(t+1)k_{t+1}(G)}{tk_t(G)} > \frac{n}{t} - \frac{n}{s} + \frac{r-s}{rs}n + \alpha n = \left( \frac{r-t}{rt} + \alpha \right) n.$$

Multiplying these inequalities for  $t = s+1, \dots, r$ , we obtain

$$\frac{(r+1)k_{r+1}(G)}{(s+1)k_{s+1}(G)} > n^{r-s} \prod_{t=s+1}^r \left( \frac{r-t}{rt} + \alpha \right).$$

Appealing to (4), this implies that

$$(r+1)k_{r+1}(G) > n^{r+1} \prod_{t=1}^r \left( \frac{r-t}{rt} + \alpha \right) = \alpha n^{r+1} \prod_{t=1}^{r-1} \left( \frac{r-t}{rt} + \alpha \right) \geq \alpha r^2 \left( \frac{n}{r} \right)^{r+1},$$

as required.  $\square$

## 2.3 Proof of Theorem 2

Set

$$\alpha = \frac{\mu}{n} - 1 + \frac{1}{r-1}.$$

Clearly we may assume that  $\alpha > 0$ , since otherwise the assertion is trivial. Suppose that

$$sk_s(G) > n^s \prod_{t=1}^{s-1} \left( \frac{r-t}{rt} + \alpha \right) \tag{10}$$

for some  $s \in [2, r]$ . Then, by Theorem 3,

$$(r+1)k_{r+1}(G) > \alpha \frac{r^2}{r+1} \left( \frac{n}{r} \right)^{r+1} \geq \alpha \frac{r(r-1)}{r+1} \left( \frac{n}{r} \right)^{r+1},$$

completing the proof. Thus we may and shall assume that (10) fails for every  $s \in [r-1]$ .

From Theorem 1 we have

$$(r+1)k_{r+1}(G) \geq \mu^{r+1}(G) - \sum_{s=2}^r (s-1)k_s(G)\mu^{r+1-s}(G). \quad (11)$$

Substituting the bounds on  $k_s(G)$  into (11), and setting  $\mu = \mu(G)/n$ , we obtain

$$\begin{aligned} \frac{(r+1)k_{r+1}(G)}{n^{r+1}} &\geq \mu^{r+1} - \sum_{s=2}^r \mu^{r+1-s} \frac{s-1}{s} \prod_{t=1}^{s-1} \left( \frac{r-t}{rt} + \alpha \right) \\ &\geq \mu^{r+1} - \mu^{r+1-2} \frac{1}{2} \left( \frac{r-1}{r} + \alpha \right) + \sum_{s=3}^r \frac{s-1}{s} \mu^{r+1-s} \prod_{t=1}^{s-1} \left( \frac{r-t}{rt} + \alpha \right) \\ &\geq \mu^{r+1-2} \left( \mu^2 - \frac{1}{2} \left( \frac{r-1}{r} + \alpha \right) \right) + \sum_{s=3}^r \frac{s-1}{s} \mu^{r+1-s} \prod_{t=1}^{s-1} \left( \frac{r-t}{rt} + \alpha \right) \\ &\geq \mu^{r+1-2} \left( \frac{r-1}{r} + \alpha \right) \left( \frac{r-2}{2r} + \alpha \right) + \sum_{s=3}^r \frac{s-1}{s} \mu^{r+1-s} \prod_{t=1}^{s-1} \left( \frac{r-t}{rt} + \alpha \right). \end{aligned}$$

By induction on  $k$  we prove that, for all  $k = 2, \dots, r$ ,

$$\frac{(r+1)k_{r+1}(G)}{n^{r+1}} \geq \mu^{r+1-k} \prod_{t=1}^k \left( \frac{r-t}{rt} + \alpha \right) - \sum_{s=k+1}^r \frac{s-1}{s} \mu^{r+1-s} \prod_{t=1}^{s-1} \left( \frac{r-t}{rt} + \alpha \right)$$

and hence,

$$\frac{(r+1)k_{r+1}(G)}{n^{r+1}} \geq \mu \prod_{t=1}^r \left( \frac{r-t}{rt} + \alpha \right) \geq \alpha \frac{r-1}{r} \prod_{t=1}^{r-1} \frac{r-t}{rt} = \alpha \frac{r-1}{r^r}.$$

It follows that

$$k_{r+1}(G) \geq \alpha \frac{r(r-1)}{r+1} \left( \frac{n}{r} \right)^{r+1},$$

as required.  $\square$

## 2.4 Proof of Theorem 4

Inequality (1) for  $s = 2$  together with (6) implies that

$$2 \frac{r-1}{r} e(G) \geq \mu^2(G) > \left( \frac{r-1}{r} - \alpha \right)^2 n^2 > \left( \left( \frac{r-1}{r} \right)^2 - 2\alpha \frac{r-1}{r} \right) n^2,$$

and so,

$$e(G) \geq \left( \frac{r-1}{2r} - 2\alpha \right) n^2.$$

To complete our proof, let us recall the following stability theorem proved by Nikiforov and Rousseau in [11]. Let  $r \geq 2$  and  $0 < \beta \leq 2^{-9}r^{-6}$ , and let  $G = G(n)$  be a  $K_{r+1}$ -free graph satisfying

$$e(G) \geq \left( \frac{r-1}{2r} - \beta \right) n^2.$$

Then  $G$  contains an induced  $r$ -partite graph  $G_0$  of order  $v(G_0) > (1 - 2\alpha^{1/3})n$  and with minimum degree

$$\delta(G_0) \geq \left( 1 - \frac{1}{r} - 4\beta^{1/3} \right) n.$$

Setting  $\beta = 2\alpha$ , in view of  $4 \cdot 2^{1/3} < 6$ , the required inequalities follow.  $\square$

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